

Large-scale superfluid vortex rings at nonzero temperatures

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We numerically model experiments in which large-scale vortex rings – bundles of quantized vortex loops – are created in superfluid helium by a piston-cylinder arrangement. We show that the presence of a normal fluid vortex ring together with the quantized vortices is essential to explain the coherence of these large-scale vortex structures at nonzero temperatures, as observed experimentally. Finally we argue that the interaction of superfluid and normal fluid vortex bundles is relevant to recent investigations of superfluid turbulence.

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I. INTRODUCTION

Quantized vortex rings in superfluid liquid helium (helium II) have a special place [1] in the vast vortex dynamics literature, and have been recently visualized directly using tracer particles[2]. The quantization of the circulation and the infinitesimal thickness of the vortex core make superfluid rings real-life examples of the (otherwise idealized) inviscid Euler dynamics which is described in textbooks. For example, a well-known textbook result [3] is that the self-induced translational velocity v_{self} of a thin-core vortex ring of large radius R , vortex core radius $a_0 \ll R$ and circulation Γ , moving in an incompressible inviscid fluid, is inversally proportional to R , and has the form:

$$v_{self} = \frac{\Gamma}{4\pi R} \left[\ln \left(\frac{8R}{a_0} \right) - \frac{1}{2} \right]. \quad (1)$$

Eq. (1) has been verified [4] in superfluid helium. It must be noticed that in the helium context Γ and a_0 are not arbitrary parameters, but are fixed by quantum mechanical constraints [5]: $\Gamma = \kappa = h/m \approx 10^{-3} \text{ cm}^2/\text{s}$ (called the quantum of circulation, where h is Planck's constant and m the mass of one ^4He atom), and $a_0 \approx 10^{-8} \text{ cm}$ (proportional to the superfluid coherence length).

In superfluid liquid helium, vortex rings are usually created by high voltage tips. The tips discharge ions which are accelerated by imposed electric fields, until, upon reaching a critical velocity, quantized vorticity is nucleated [6–8]. The radius R of the resulting rings is rather small, typically of the order of one micrometre only [9].

The more traditional piston–cylinder arrangement which is used to create much larger vortex rings in water or air has been implemented in liquid helium only few times, but with remarkable results. Borner *et al.* [10–12] pushed helium through the circular hole a cylindrical tube (diameter $D = 0.8 \text{ cm}$) by the single stroke of a piston. Using acoustic methods, they determined size, position, velocity and circulation of the localised superfluid vortex structure which was ejected from the tube's hole. They found that the structure travels a distance which is at least 10 times its diameter $2R$ with reproducibly constant velocity and circulation. They observed that the circulation ($2.3 < \Gamma < 4.85 \text{ cm}^2/\text{s}$ depending on the piston's action) is much larger than κ . One infers that a large-scale macroscopic vortex ring was created, consisting of a large number ($N = \Gamma/\kappa \approx 10^3$) of individual, concentric, coaxial superfluid vortex rings travelling in the same direction – essentially a compact bundle of rings. This interpretation was strengthened by further experiments [13, 14], particularly by Murakami and collaborators [15] who directly visualized the bundle using frozen hydrogen–deuterium particles.

In a recent paper [16], we discovered that bundles of rings moving in a perfect Euler fluid travel coherently over relatively large distances compared to their size. During the evolution, the individual rings within a bundle move around each other in a leapfrogging fashion. Reconnections between individual rings do not seem to affect the speed and the coherence of the bundle (in other words, laminar and turbulent bundles move at approximately the same speed). However, this work did not account for the temperature independence observed by Borner *et al.* over the explored temperature range $1.3 < T < 2.15 \text{ K}$. At nonzero temperatures helium II consists of two independent interpenetrating fluids: the inviscid superfluid (with density ρ_s) and the viscous normal fluid (with density ρ_n , where $\rho = \rho_s + \rho_n$ is the total density), which affects the superfluid vortices via friction force. At $T = 1.3 \text{ K}$ the normal fluid is certainly negligible [17] ($\rho_n/\rho = 4.5\%$), but at $T = 2.15 \text{ K}$ it is certainly not ($\rho_n/\rho = 87.2\%$). Therefore the friction [18] should rapidly destroy a vortex ring over a short distance. In fact, the range Δz of a vortex ring of initial radius R is [18]

$$\Delta z \approx \frac{\rho_s \kappa}{\gamma} \frac{(1 - \gamma \gamma_0 / (\rho_s^2 \kappa^2))}{(1 - \gamma'_0 / (\rho_s \kappa))} R, \quad (2)$$

where γ_0 , γ'_0 and γ are temperature-dependent [18] friction coefficients. Taking for example $R = 0.0896 \text{ cm}$, we find that $\Delta z/R \approx 13.4, 7.8, 4.7, 2.1$ and 1.3 respectively at $T = 1.3, 1.5, 1.7, 1.9, 2.1$ and 2.15 K , in contradiction with the experiments that $\Delta z/R > 20$.

The aim of this paper is therefore to go beyond the $T = 0$ calculation of Ref. [16], and explore the effect of the friction.

II. METHOD

Following the method of Schwarz [19], we model vortex lines as space curves $\mathbf{s}(\xi, t)$ of circulation κ (where t is time and ξ is arc length) which move according to

$$\frac{ds}{dt} = \mathbf{v}_s + \alpha \mathbf{s}' \times (\mathbf{v}_n - \mathbf{v}_s) - \alpha' \mathbf{s}' \times [\mathbf{s}' \times (\mathbf{v}_n - \mathbf{v}_s)]. \quad (3)$$

where α and α' are temperature-dependent friction coefficients [17, 18], \mathbf{v}_n is the normal fluid velocity, $\mathbf{s}' = d\mathbf{s}/d\xi$ is the unit tangent vector to the line at \mathbf{s} , and

$$\mathbf{v}_s(\mathbf{s}) = -\frac{\kappa}{4\pi} \oint_{\mathcal{L}} \frac{(\mathbf{s} - \mathbf{r})}{|\mathbf{s} - \mathbf{r}|^3} \times d\mathbf{r}, \quad (4)$$

where the line integral extends over the entire vortex configuration \mathcal{L} . The technique to discretize the vortex lines, desingularize the Biot–Savart integral (4) and perform vortex reconnections when two vortex lines come very close to each other has been already described in the literature [20–22].

The initial condition at $t = 0$ is a bundle of N coaxial concentric vortex rings set within a torus of small radius a and large radius R . For large N , the rings are arranged so that they form a triangular lattice on the cross-section of the torus [16]. This choice constraint the choice of N to “hexagonal” numbers $N = 1, 3, 7, 19, \dots$.

In all numerical simulations, the distance between the discretization points along vortex lines is algorithmically kept between the values $\Delta\xi$ and $\Delta\xi/2$, where $\Delta\xi$ is the parameter which determines the numerical resolution. Typically we set $\Delta\xi$ so that $\ell/\Delta\xi \approx 10$ where ℓ is the intervortex distance; for example, results for the $N = 3$ bundle described below are obtained for $R = 0.06$ cm, $\ell = 0.015$ cm and $\Delta\xi = 0.00149$ cm, but we have done checks with smaller values of $\Delta\xi$, e.g. $\Delta\xi = 0.00075$ cm, and found that the velocity of the bundle does not change.

The computational cost of the Biot–Savart integrals (4) scales with the square of the number of discretization points; an increase of discretization points also requires a smaller time step. Thus practical computing limitations prevent us from simulating bundles with large N .

III. RESULTS

First we recover the previous results [16] corresponding to $T = 0$ (no friction). The governing equation is Eq. (3) with $\alpha = \alpha' = 0$. For example, Fig. (1) shows the stable, periodic leapfrogging motion of $N = 3$ vortex rings around each other. In turn, the vortex ring at the back of the bundle shrinks in size and goes inside the other two rings, speeding up and going ahead of them; then it grows in size, slows down, and goes above and around the other rings, falling behind them. This motion (a generalization of the well-known leapfrogging of two vortex rings) repeats in periodic fashion. It is interesting to notice that the vortex rings remain circular – no Kelvin waves develop.

Secondly, we study the effect of the friction on the motion of the bundles. The governing equation is Eq. (3) with nonzero α and α' but with $\mathbf{v}_n = 0$ (normal fluid at rest). We find that, in general, a vortex bundle at finite temperatures in such stationary normal fluid tends to lose its constituent vortex rings one by one, until it vanishes. To illustrate this effect, we choose the same $N = 3$ vortex configuration of Fig. (1), and show what happens to it at $T = 2.02$ K in Fig. (2). The ring which is inside the other two rings tends to shrink too much, speeding up and going ahead of them, then escaping the bundle and vanishing away.

Larger bundles ($N > 3$) suffer the same fate, which is not consistent with Borner’s observation that bundles travel in a stable way for a distance much larger than their size. The question is: what stabilizes the bundles in the experiments?

We argue that the answer is the viscous normal fluid, which is not at rest, but is pushed out of the hole together with the superfluid, and must roll up in the classical form of an ordinary vortex ring. In fact Borner *et al.* measured the circulation of this normal fluid ring and found that it is the same circulation of the superfluid vortex bundle. To model the normal fluid ring, we define the following toroidal, Rankine vortex flow \mathbf{v}_n . At each instant, let the origin of our reference frame be the (moving) centre of the superfluid vorticity and z be the direction of propagation of the bundle. In each cross section of the torus, let r be the radial distance away from the axis around the torus. The normal fluid velocity \mathbf{v}_n has two components. The first component is a uniform flow along z with the same speed of the centre of the superfluid vorticity. The second component, tangential around the torus, is equal to a uniform solid body rotation (proportional to r) inside the torus, and an irrotational flow (inversally proportional to r) with circulation $N\kappa$ outside the torus [24]. To accommodate distortions of the core of superfluid bundle (which is initially circular, but tends to become D-shaped during the evolution) and the dissipative nature of the normal fluid vortex ring (which we expect to spreads spatially), the transition from solid body rotation and irrotational flow is not at $r = a$, but at $r = 2a$.

The model is simple but gives results in agreement with the experiments. Fig. (3) shows the same $N = 3$ superfluid vortex bundle at $T = 2.02$ K with \mathbf{v}_n defined as above. It is apparent that the bundle moves in a stable way.

The result holds true for bundles with $N > 3$. Table (I) summarizes the initial conditions of our numerical simulations, listing the large radius R and the small radius a of the torus, their ratio R/a , the initial vortex length λ_0 (which is not exactly equal to $2\pi RN$ because the rings within the torus have different radii), the initial average curvature (which, for the same reason, is not exactly equal to $1/R$), and the initial intervortex distance ℓ_0 . These initial conditions are chosen to best fit Borner’s experiment with the same ℓ_0 , see the discussion in Ref. [16].

Tables (II) and (III) show the results of our numerical simulations respectively at $T = 0$ (no friction) and $T = 2.02$ K (with friction and normal fluid vortex ring \mathbf{v}_n). The simulations are stopped when a vortex bundle has travelled a distance of the order of 10 diameters, as observed in the experiments. The tables list the time t at which we stop the simulation, the distance travelled Δz by the centre of vorticity of each bundle in terms of the initial diameter $2R$, the vortex length Λ at time t in terms of the initial vortex length Λ_0 , the average curvature \bar{c} at time t in terms of the initial average curvature \bar{c}_0 , and the speed v of the vortex bundle at time t in terms of the initial speed v_0 .

The main result is that, in the presence of the normal fluid vortex ring, a superfluid bundle travels a significant distance despite the friction, in agreement with the experiment of Borner *et al.* .

We find that, with or without friction, the smaller bundles (small N) tend to remain circular (at least within the time-scales of our simulations); the larger bundles (large N) develop instabilities which induce vortex reconnections, changing the actual number of vortex rings in the bundle, and causing Kelvin waves. Interestingly, vortex reconnections have no significant effect on the coherence of a bundle – a turbulent vortex bundle only seems to travel somewhat slower.

More precisely, the $N = 7$ bundle has the first reconnection at $t = 26.98$ s for $T = 0$ and at about the same time ($t = 24.06$ s) for $T = 2.02$ K. The fact that the first reconnection in the evolution of the $N = 19$ bundle at $T = 0$ occurs much sooner ($t = 19.89$ s) than at $T = 2.02$ K ($t = 45.76$ s).

The main difference between the appearance of the vortex bundles at $T = 0$ and $T = 2.02$ K is the wavelengths which seem excited: low temperatures favour short waves, high temperatures long waves, as shown in Figs. (4) and (5) respectively. The increase of average curvature reflects this effect. This is consistent with the observation made by Tsubota *et al.* [25] that, at low temperatures, turbulent vortex tangles tend to be more wiggly.

IV. CONCLUSION

Practical computer limitations prevents us from studying vortex bundles with thousands of rings as in the experiments. Nevertheless, exploring in detail what happens to small bundles gives us insight into the physics of the problem.

Our results suggest that bundles of superfluid vortex rings can travel coherently a significant distance, at least one order of magnitude larger than their diameter, in agreement with experimental observations. The effect seems temperature independent. The normal fluid vortical structure which is generated by the piston-cylinder set up has been observed to move along the superfluid vortical structure. The coexistence of superfluid and normal fluid structures effectively inhibits the friction between the superfluid vortices and the normal fluid from dissipating the bundles, and this effect explains the experimental observation of Borner *et al.* [10–12] that large-scale vortex rings remain stable at nonzero temperatures.

To put the large-scale vortex ring experiments in a wider context, it is worth recalling a related problem: the “spontaneous” appearance of bundles of vortices in superfluid turbulence (opposed to the “forced” generation of bundles of vortices by the piston-cylinder arrangement described here). Missing any direct experimental observation, the existence and the non-existence of such bundles [26–30] or of partial polarization of vortex lines [31, 32], has been discussed in the literature, particularly with respect to the Kolmogorov spectrum [33, 34]. In this context, the vortex rings generated by the piston-cylinder setup provide a “forced” but controlled method to study the coupling of normal fluid and superfluid.

Finally, we notice that the detailed mechanism of generation of the double (normal fluid and superfluid) vortex ring structure at the hole of the cylinder is an interesting problem of two-fluid hydrodynamics which would be worth studying.

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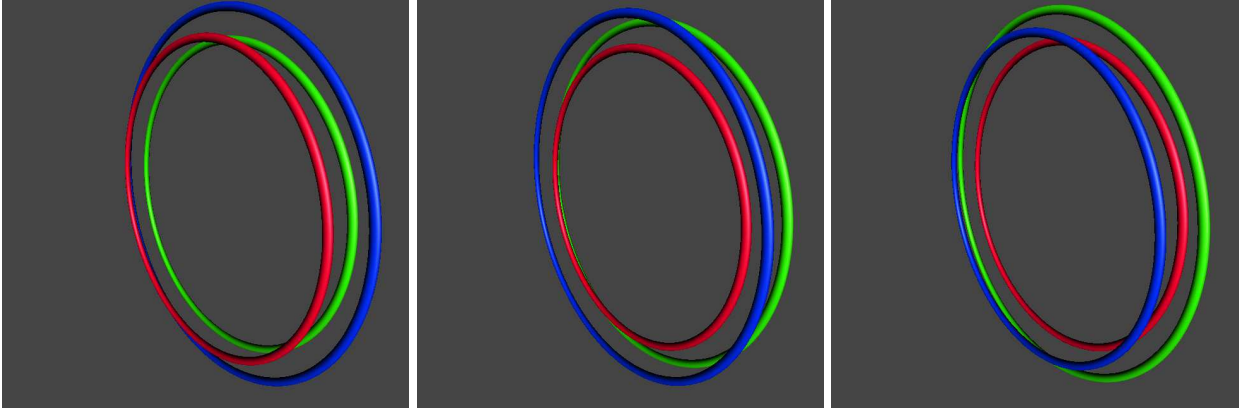


FIG. 1: (Color online). Motion of 3 vortex rings (initial condition: $R = 0.0896$ cm, $a = 0.0075$ cm) at zero temperature (no friction). Left: $t = 0$; middle: $t = 3.6$; right: $t = 7.2$ s. The vortex bundle is stable, and each ring leapfrogs around the others. Each vortex line is represented by a tube[23] whose thickness is for the sake of visualization only (vortex lines have infinitesimal thickness).

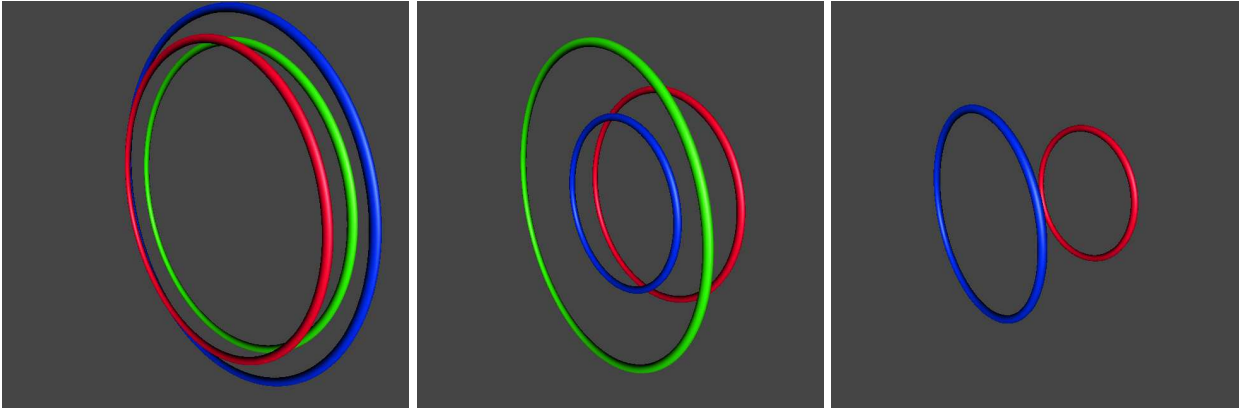


FIG. 2: (Color online). Motion of 3 vortex rings in the presence of friction at $T = 2.02$ K with $\mathbf{v}_n = 0$. The initial condition is the same as in Fig. (1). Left: $t = 0$; middle: $t = 3.6$; right: $t = 7.2$ s. The vortex bundle decays: one by one, all vortex rings shrink and vanish on the axis of propagation.

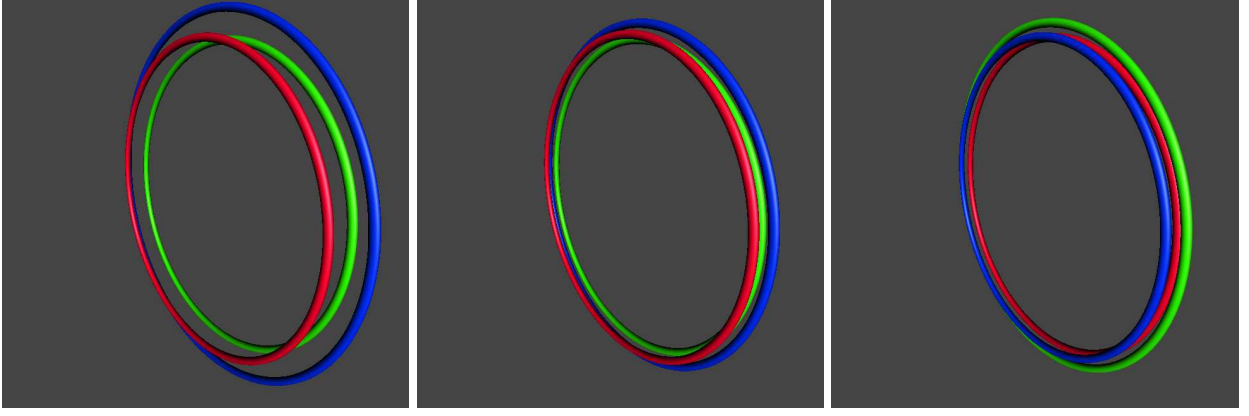


FIG. 3: (Color online). Motion of 3 vortex rings in the presence of friction at $T = 2.02$ K and normal fluid ring ($\mathbf{v}_n \neq 0$). The initial condition is the same as in Fig. (1). Left: $t = 0$; middle: $t = 3.6$; right: $t = 7.2$ s. It is apparent that the superfluid vortex bundle moves in a stable way as in the absence of friction (Fig. (1)); the individual rings leapfrog around each other.

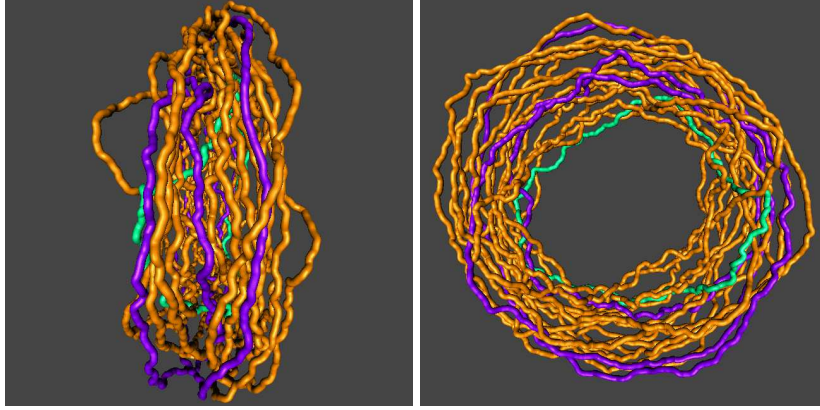


FIG. 4: (Color online). Vortex bundle at $T = 0$ (no friction). Top: Side (left) and rear (right) view of vortex bundle with $N = 19$ rings at time $t = 40$ s.

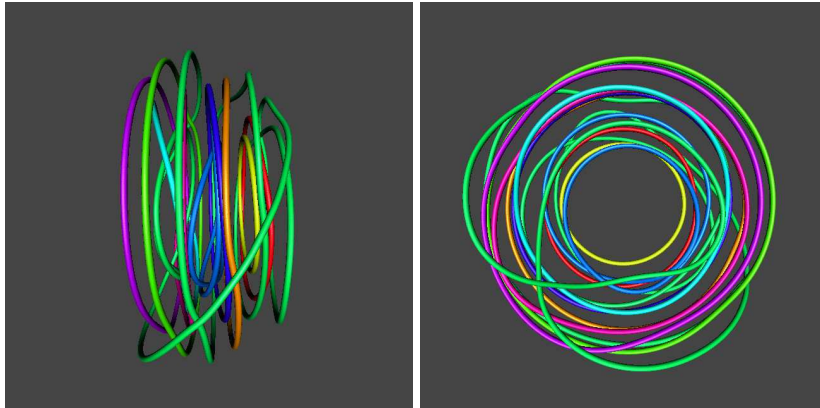


FIG. 5: (Color online). Vortex bundle at $T = 2.02$ K in the presence of friction and normal fluid ring \mathbf{v}_n . Top: Side (left) and rear (right) view of vortex bundle with $N = 19$ rings at time $t = 80$ s.

N	R cm	a cm	R/a	Λ_0 cm	\bar{c}_0 cm^{-1}	ℓ_0 cm	v_0 cm/s
2	0.06	0.075	8.0	0.754	17.19	0.015	0.0279
3	0.06	0.00866	6.92	1.131	17.11	0.015	0.0310
7	0.06	0.015	4.0	2.639	18.08	0.015	0.0450
19	0.12	0.013	4.0	14.326	10.87	0.015	0.0471

TABLE I: Initial conditions.

N	t s	$\Delta z/(2R)$	Λ/Λ_0	\bar{c}/\bar{c}_0	v/v_0
2	50	11.11	1.0	0.97	0.92
3	40	10.24	1.0	1.0	1.0
7	30	10.91	1.1	3.8	0.84
19	60	10.38	1.6	18.4	0.59

TABLE II: Evolution at $T = 0$ (no friction).

N	t s	$\Delta z/D$	Λ/Λ_0	\bar{c}/\bar{c}_0	v/v_0
2	50	11.87	0.97	1.02	0.99
3	40	11.37	0.96	1.07	1.10
7	30	10.13	1.52	1.18	0.69
19	60	10.21	0.80	1.08	0.64

TABLE III: Evolution at $T = 2.02$ K in the presence of friction and \mathbf{v}_n .